

Sobolev spaces, fine gradients and quasicontinuity on quasiopen sets in \mathbf{R}^n and metric spaces

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Abstract. We study different definitions of Sobolev spaces on quasiopen sets in a complete metric space X equipped with a doubling measure supporting a p -Poincaré inequality with $1 < p < \infty$, and connect them to the Sobolev theory in \mathbf{R}^n . In particular, we show that for quasiopen subsets of \mathbf{R}^n the Newtonian functions, which are naturally defined in any metric space, coincide with the quasicontinuous representatives of the Sobolev functions studied by Kilpeläinen and Malý in 1992. As a by-product, we establish the quasi-Lindelöf principle of the fine topology in metric spaces and study several variants of local Newtonian and Dirichlet spaces on quasiopen sets.

Key words and phrases: Dirichlet space, fine gradient, fine topology, metric space, minimal upper gradient, Newtonian space, quasicontinuous, quasi-Lindelöf principle, quasiopen, Sobolev space.

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1. Introduction

We study different definitions of Sobolev functions on nonopen sets in metric spaces. Even in \mathbf{R}^n , it is not obvious how to define Sobolev spaces on nonopen subsets, but fruitful definitions have been given on quasiopen sets U , i.e. on sets which differ from open sets by sets of arbitrarily small capacity. Kilpeläinen–Malý [18] gave the first definition of $W^{1,p}(U)$ in 1992 by means of quasicovering patches of global Sobolev functions. They also defined a generalized so-called fine gradient for functions in $W^{1,p}(U)$. More recently, Sobolev spaces, and in particular the so-called Newtonian spaces, have been studied on metric spaces. Thus by considering U as a metric space in its own right (and forgetting the ambient space) we get another candidate, the Newtonian space $N^{1,p}(U)$.

The purpose of this paper is to show that the theory of Sobolev functions nicely extends to quasiopen sets in the metric setting. In particular, we show that for quasiopen sets $U \subset \mathbf{R}^n$, the two spaces $W^{1,p}(U)$ and $N^{1,p}(U)$ coincide, with equal

norms. To be precise, the functions in $N^{1,p}(U)$ are more exactly defined than a.e., and we have the following result.

Theorem 1.1. *Let $U \subset \mathbf{R}^n$ be quasiopen, and let $u : U \rightarrow \overline{\mathbf{R}} := [-\infty, \infty]$ be an everywhere defined function. Then $u \in N^{1,p}(U)$ if and only if $u \in W^{1,p}(U)$ and u is quasicontinuous. Moreover, in this case $\|u\|_{N^{1,p}(U)} = \|u\|_{W^{1,p}(U)}$.*

On open sets in \mathbf{R}^n , the equality between the Newtonian and Sobolev spaces was proved by Shanmugalingam [23]. The proof of Theorem 1.1 is quite involved, and we will need most of the results in this paper to deduce it. We will also use several results related to the fine topology from our earlier papers [5]–[7].

In \mathbf{R}^n (and in the setting of this paper), quasiopen sets appear whenever truncations of Sobolev functions are considered. This is so because quasiopen sets coincide with the superlevel sets of suitable (quasicontinuous) representatives of the global Sobolev functions, see Fuglede [12], Kilpeläinen–Malý [18] and Proposition 4.8 below. On the other hand, it is natural to study Sobolev spaces on quasiopen sets, as these sets have enough structure to carry reasonable families of Sobolev functions, and in particular of test functions. This is important for studying partial differential equations and variational problems on such sets, see [18].

The metric space approach to Sobolev functions makes it in principle possible to consider Sobolev spaces and variational problems on arbitrary sets, but it turns out that there is not much point in considering more general sets than the quasiopen ones. More precisely, in Björn–Björn [5] it was shown that the Dirichlet problem for p -harmonic functions in an arbitrary set coincides with the one in the set’s fine interior. Moreover, the spaces of Sobolev test functions with zero boundary values are the same for the set and its fine interior. Even in the metric setting of this paper, finely open sets are quasiopen and quasiopen sets differ from finely open sets only by sets of capacity zero [7]. The results in [5] also show that restrictions of (upper) gradients from the underlying space (such as \mathbf{R}^n) behave well on quasiopen sets, but not on more general sets.

Quasiopen sets can also be regarded as a link between Sobolev spaces and potential theory, in which finely open sets play an important role. Finely open sets form the fine topology, which is the coarsest topology making all superharmonic functions continuous [6] and serves as a tool for many deep properties in potential theory. Finely open, and thus quasiopen, sets can be very different from the usual open sets. The simplest examples of quasiopen sets are open sets with an arbitrary set of zero capacity removed or added. Such a removed set can be dense, causing the interior of the resulting set to be empty. From the point of view of potential theory, such a set behaves like the original open one.

A typical finely open set is the complement of the Lebesgue spine in \mathbf{R}^3 with the tip of the spine added to it. This is natural, since for harmonic and superharmonic functions, the tip behaves more like an interior point than a boundary point. More generally, finely open sets contain points which have only a “thin” connection with the complement of the set. Examples 9.5 and 9.6 in [5] describe a closed set $E \subset \mathbf{R}^n$ with positive Lebesgue measure and empty Euclidean interior, but whose fine interior has full measure in E and is thus suitable for solving the Dirichlet problem on it. The set looks like a Swiss cheese and its complement consists of countably many balls of shrinking radii. We refer to the introductions in [5]–[7], and the references therein, for discussions on the fine topology and the history of the (fine) nonlinear potential theory.

In this paper we assume that X is a complete metric space equipped with a doubling measure supporting a p -Poincaré inequality, $1 < p < \infty$. As hinted before Theorem 1.1, Newtonian functions are better representatives than the usual Sobolev functions. Namely, it was shown by Björn–Björn–Shanmugalingam [9], that all Newtonian functions on open sets are quasicontinuous. Moreover, they are

finely continuous outside of sets of zero capacity, by J. Björn [11] or (independently) Korte [19]. We extend both these results to quasiopen sets in Section 6:

Theorem 1.2. *Let $U \subset X$ be quasiopen, and $u \in N^{1,p}(U)$. Then u is quasicontinuous in U and finely continuous quasieverywhere in U .*

This will be used as an important tool when establishing Theorem 1.1 in Section 7. Another tool that we will need for proving both theorems above is the quasi-Lindelöf principle of the fine topology, which we obtain in Section 4. Along the way, in Section 5, we introduce and study several variants of local Newtonian spaces on quasiopen sets, some of them inspired by the earlier definitions in \mathbf{R}^n given by Kilpeläinen–Malý [18] and Malý–Ziemer [22]. In fact, Theorem 1.2 holds for these local spaces as well, see Theorem 6.1. They can be used as natural spaces for considering p -harmonic and superharmonic functions on quasiopen sets in metric spaces, as in [18], even though we will not pursue this line of research here. These local spaces are also suitable for defining the so-called p -fine upper gradients which have been tailored for the proof of Theorem 1.1 but turn out to be a useful generalization of the minimal p -weak upper gradients as well, see Theorems 7.3 and 7.4.

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2. Notation and preliminaries

In Sections 2 and 3 we assume that $1 \leq p < \infty$, while in later sections we will assume that $1 < p < \infty$. Starting from Section 4 we will also assume that X is complete and supports a p -Poincaré inequality, and that μ is doubling, see the definitions below.

We assume throughout the paper that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all (open) balls $B \subset X$. It follows that X is separable, see Proposition 1.6 in Björn–Björn [4]. The σ -algebra on which μ is defined is obtained by the completion of the Borel σ -algebra. We also assume that $\Omega \subset X$ is a nonempty open set.

We say that μ is *doubling* if there exists $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$0 < \mu(2B) \leq C\mu(B) < \infty.$$

Here and elsewhere we let $\lambda B = B(x_0, \lambda r)$. A metric space with a doubling measure is proper (i.e. closed and bounded subsets are compact) if and only if it is complete.

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds . We follow Heinonen and Koskela [15] in introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all nonconstant, compact and rectifiable curves $\gamma : [0, l_\gamma] \rightarrow X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds, \quad (2.1)$$

where we follow the convention that the left-hand side is ∞ whenever at least one of the terms therein is infinite.

If g is a nonnegative measurable function on X and if (2.1) holds for p -almost every curve (see below), then g is a p -weak upper gradient of f .

Here we say that a property holds for p -almost every curve if it fails only for a curve family Γ with zero p -modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a p -weak upper gradient need not be a Borel function, it is only required to be measurable. On the other hand, every measurable function g can be modified on a set of measure zero to obtain a Borel function, from which it follows that $\int_\gamma g ds$ is defined (with a value in $[0, \infty]$) for p -almost every curve γ . For proofs of these and all other facts in this section we refer to Björn–Björn [4] and Heinonen–Koskela–Shanmugalingam–Tyson [16].

The p -weak upper gradients were introduced in Koskela–MacManus [20]. It was also shown there that if $g \in L^p_{\text{loc}}(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j - g \rightarrow 0$ in $L^p(X)$. If f has an upper gradient in $L^p_{\text{loc}}(X)$, then it has a *minimal p -weak upper gradient* $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every p -weak upper gradient $g \in L^p_{\text{loc}}(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [24]. The minimal p -weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p_{\text{loc}}(X)$. Following Shanmugalingam [23], we define a version of Sobolev spaces on the metric measure space X .

Definition 2.2. Let for measurable f ,

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [23]. We also define

$$D^p(X) = \{f : f \text{ is measurable and has an upper gradient in } L^p(X)\}.$$

In this paper we assume that functions in $N^{1,p}(X)$ and $D^p(X)$ are defined everywhere (with values in $\overline{\mathbf{R}} := [-\infty, \infty]$), not just up to an equivalence class in the corresponding function space.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $f \in N^{1,p}_{\text{loc}}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$. The spaces $D^p(E)$ and $D^p_{\text{loc}}(E)$ are defined similarly. For a function $f \in D^p_{\text{loc}}(E)$ we denote the minimal p -weak upper gradient of f with respect to E by $g_{f,E}$.

Definition 2.3. The *Sobolev capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E .

The Sobolev capacity is countably subadditive. We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has Sobolev capacity zero. The Sobolev capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$

q.e. Moreover, Corollary 3.3 in Shanmugalingam [23] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

Capacity is also important for the following two notions, which are central in this paper.

Definition 2.4. A set $U \subset X$ is *quasiopen* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $G \cup U$ is open.

A function u defined on a set $E \subset X$ is *quasicontinuous* if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{E \setminus G}$ is finite and continuous.

The quasiopen sets do not in general form a topology, see Remark 9.1 in Björn–Björn [5]. However it follows easily from the countable subadditivity of C_p that countable unions and finite intersections of quasiopen sets are quasiopen. (We consider finite sets to be countable throughout the paper.) For characterizations of quasiopen sets and quasicontinuous functions see Björn–Björn–Malý [8] and Proposition 4.8.

Together with the doubling property defined above, the following Poincaré inequality is often a standard assumption on metric spaces.

Definition 2.5. We say that X supports a *p-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions f on X and all upper gradients g of f ,

$$\int_B |f - f_B| d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p}, \quad (2.2)$$

where $f_B := \int_B f d\mu / \mu(B)$.

In the definition of Poincaré inequality we can equivalently assume that g is a p -weak upper gradient. In \mathbf{R}^n equipped with a doubling measure $d\mu = w dx$, where dx denotes Lebesgue measure, the p -Poincaré inequality (2.2) is equivalent to the *p-admissibility* of the weight w in the sense of Heinonen–Kilpeläinen–Martio [14], see Corollary 20.9 in [14] and Proposition A.17 in [4].

If X is complete and supports a p -Poincaré inequality and μ is doubling, then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [23]. Moreover, all functions in $N^{1,p}(X)$ and those in $N^{1,p}(\Omega)$ are quasicontinuous, see Björn–Björn–Shanmugalingam [9]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [14, p. 96], see [4, Appendix A.2] for a proof of this fact valid in weighted \mathbf{R}^n . This is the main reason why, unlike in the classical Euclidean setting, we do not need to require the functions competing in the definitions of capacity to be 1 in a neighbourhood of E . For recent related progress on the density of Lipschitz functions see Ambrosio–Colombo–Di Marino [1] and Ambrosio–Gigli–Savaré [2].

In Section 4 the fine topology is defined by means of thin sets, which in turn use the variational capacity cap_p . To be able to define the variational capacity we first need a Newtonian space with zero boundary values. We let, for an arbitrary set $A \subset X$,

$$N_0^{1,p}(A) = \{f|_A : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus A\}.$$

One can replace the assumption “ $f = 0$ on $X \setminus A$ ” with “ $f = 0$ q.e. on $X \setminus A$ ” without changing the obtained space $N_0^{1,p}(A)$. Functions from $N_0^{1,p}(A)$ can be extended by zero in $X \setminus A$ and we will regard them in that sense if needed.

Definition 2.6. The *variational capacity* of $E \subset \Omega$ with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_X g_u^p d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E .

3. Quasiopen and p -path open sets

In this section we assume that $1 \leq p < \infty$, but μ is not required to be doubling and, apart from Proposition 3.5, no Poincaré inequality is required.

Recall that quasiopen sets were defined in Definition 2.4. The following is another useful notion in connection with Sobolev functions. It was introduced by Shanmugalingam [24].

Definition 3.1. A set $U \subset X$ is p -path open if for p -almost every curve $\gamma : [0, l_\gamma] \rightarrow X$, the set $\gamma^{-1}(U)$ is (relatively) open in $[0, l_\gamma]$.

Lemma 3.2. (Shanmugalingam [24, Remark 3.5] and Björn–Björn [5, Lemma 9.3]) *If U is quasiopen, then U is p -path open and measurable.*

The following lemma is from [5, Corollary 3.7]. Recall that $g_{u,U}$ denotes the minimal p -weak upper gradient of u with respect to U , while g_u denotes the minimal p -weak upper gradient of u with respect to X .

Lemma 3.3. *Let U be a measurable p -path open set and $u \in D_{\text{loc}}^p(X)$. Then*

$$g_{u,U} = g_u \quad \text{a.e. in } U.$$

In particular, this holds if U is quasiopen.

For a family of curves Γ on X , we define its p -modulus

$$\text{Mod}_p(\Gamma) := \inf \int_X \rho^p d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho ds \geq 1$ for all $\gamma \in \Gamma$. Let Γ_E^U be the set of curves $\gamma : [0, l_\gamma] \rightarrow U$ which hit $E \subset U$, i.e. $\gamma^{-1}(E) \neq \emptyset$.

We are now ready to make some new observations for p -path open sets.

Lemma 3.4. *If U is p -path open and $E \subset U$, then $\text{Mod}_p(\Gamma_E^U) = \text{Mod}_p(\Gamma_E^X)$.*

If U is not p -path open, then this is not true in general: Consider, e.g., $E = \{0\} \subset \mathbf{R}$ and $U = \mathbf{Q}$, in which case $\text{Mod}_p(\Gamma_E^U) = 0 < \text{Mod}_p(\Gamma_E^{\mathbf{R}})$.

Proof. Since $\Gamma_E^U \subset \Gamma_E^X$, we have $\text{Mod}_p(\Gamma_E^U) \leq \text{Mod}_p(\Gamma_E^X)$.

Conversely, as U is p -path open, p -almost every curve $\gamma \in \Gamma_E^X$ is such that $\gamma^{-1}(U)$ is relatively open in $[0, l_\gamma]$ (and we can ignore the other curves in Γ_E^X). Moreover $\gamma^{-1}(U) \supset \gamma^{-1}(E) \neq \emptyset$. Hence $\gamma^{-1}(U)$ is a nonempty countable union of relatively open intervals of $[0, l_\gamma]$ (see e.g. Lemma 1.4 in [4]), at least one of which contains a point $t \in \gamma^{-1}(E)$. We can thus find a small compact interval $[a, b] \ni t$, $0 \leq a < b \leq l_\gamma$, such that $[a, b] \subset \gamma^{-1}(U)$. Then $\gamma|_{[a,b]} \in \Gamma_E^U$, and Lemma 1.34 (c) in [4] implies that $\text{Mod}_p(\Gamma_E^X) \leq \text{Mod}_p(\Gamma_E^U)$. \square

Proposition 3.5. *Assume that X supports a p -Poincaré inequality. Let U be a measurable p -path open set and $u \in D^p(U)$. Then $C_p(\{x \in U : |u(x)| = \infty\}) = 0$.*

Proof. Let $E = \{x \in U : |u(x)| = \infty\}$. As $u \in D^p(U)$, u is measurable and thus also E is measurable. Let $g \in L^p(U)$ be an upper gradient of u in U . Each curve $\gamma \in \Gamma_E^U$ contains a nonconstant subcurve $\tilde{\gamma}$ that either starts or ends in E . As g is an upper gradient of u in U and $|u(x)| = \infty$ for $x \in E$, we see that

$$\int_\gamma g ds \geq \int_{\tilde{\gamma}} g ds = \infty.$$

Hence, $\text{Mod}_p(\Gamma_E^U) = 0$. By Lemma 3.4, $\text{Mod}_p(\Gamma_E^X) = \text{Mod}_p(\Gamma_E^U) = 0$. Finally, Proposition 4.9 in [4] shows that $C_p(E) = 0$. \square

4. Fine topology

Throughout the rest of the paper, we assume that $1 < p < \infty$, that X is complete and supports a p -Poincaré inequality, and that μ is doubling.

In this section we recall the basic facts about the fine topology associated with Sobolev spaces and prove some auxiliary results which will be crucial in the subsequent sections.

Definition 4.1. A set $E \subset X$ is *thin* at $x \in X$ if

$$\int_0^1 \left(\frac{\text{cap}_p(E \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty. \quad (4.1)$$

A set $U \subset X$ is *finely open* if $X \setminus U$ is thin at each point $x \in U$.

It is easy to see that the finely open sets give rise to a topology, which is called the *fine topology*. Every open set is finely open, but the converse is not true in general.

In the definition of thinness, we make the convention that the integrand is 1 whenever $\text{cap}_p(B(x, r), B(x, 2r)) = 0$. This happens e.g. if $X = B(x, 2r)$, but never if $r < \frac{1}{2} \text{diam } X$. Note that thinness is a local property.

Definition 4.2. A function $u : U \rightarrow \overline{\mathbf{R}}$, defined on a finely open set U , is *finely continuous* if it is continuous when U is equipped with the fine topology and $\overline{\mathbf{R}}$ with the usual topology.

Since every open set is finely open, the fine topology generated by the finely open sets is finer than the metric topology. In fact, it is the coarsest topology making all superharmonic functions finely continuous, by J. Björn [11, Theorem 4.4], Korte [19, Theorem 4.3] and Björn–Björn–Latvala [6, Theorem 1.1]. See [4, Section 11.6] and [6] for further discussion on thinness and the fine topology.

The following definition will play an important role in the later sections of the paper.

Definition 4.3. A set $E \subset A$ is a *p -strict subset* of A if there is a function $u \in N_0^{1,p}(A)$ such that $u = 1$ on E .

Equivalently, it can in addition be required that $0 \leq u \leq 1$, as in Kilpeläinen–Malý [18]. The following lemma shows that there are many nice p -strict subsets of finely open sets. They play the role of compact subsets of open sets. In particular, there is a base of fine neighbourhoods, consisting only of p -strict subsets.

Lemma 4.4. *Let V be finely open and let $x_0 \in V$. Then there exist a finely open $W \ni x_0$ and an upper semicontinuous finely continuous function $v \in N_0^{1,p}(V)$ with compact support in V such that $0 \leq v \leq 1$ everywhere and $v = 1$ in W .*

In particular, W is a p -strict subset of V and $W \Subset V$.

Recall that $W \Subset V$ if \overline{W} is a compact subset of V .

Proof. Since V is finely open, $E := X \setminus V$ is thin at x_0 . By the weak Cartan property in Björn–Björn–Latvala [6, Theorem 5.1] there exists a lower semicontinuous finely continuous function $u \in N^{1,p}(B)$ in a ball $B \ni x_0$ such that $0 < u \leq 1$ in B , $u = 1$ in $E \cap B$ and $u(x_0) < 1$. Let $0 \leq \eta \leq 1$ be a Lipschitz function with compact support in B such that $\eta(x_0) = 1$. Then $f = \eta(1 - u) \in N_0^{1,p}(V)$ is upper semicontinuous and finely continuous in X and $f(x_0) = 1 - u(x_0) > 0$.

To conclude the proof, set

$$W = \{x \in V : f(x) > \tfrac{1}{2}f(x_0)\} \quad \text{and} \quad v(x) = \min \left\{ 1, \left(\frac{4f(x)}{f(x_0)} - 1 \right)_+ \right\}.$$

A simple calculation shows that $v = 1$ in W , and the upper semicontinuity of f implies that $W \subseteq \text{supp } v \subseteq V$. As f is finely continuous, W is finely open. \square

The following quasi-Lindelöf principle will play an important role in the later sections.

Theorem 4.5. (Quasi-Lindelöf principle) *For each family \mathcal{V} of finely open sets there is a countable subfamily \mathcal{V}' such that*

$$C_p \left(\bigcup_{V \in \mathcal{V}} V \setminus \bigcup_{V' \in \mathcal{V}'} V' \right) = 0.$$

Our proof of the quasi-Lindelöf principle in metric spaces is quite similar to the proof in Heinonen–Kilpeläinen–Malý [13, Theorem 2.3] for unweighted \mathbf{R}^n . We include it here for the reader's convenience. We will need the following lemma.

Lemma 4.6. *Let $x \in X$ and let $\{x_k\}_{k=1}^\infty$ be a sequence of points in X such that $d(x_k, x) < 2^{-k-2}$ for $k = 1, 2, \dots$. If $B_k = B(x_k, 3 \cdot 2^{-k-2})$, then a set $E \subset X$ is thin at x if and only if*

$$\sum_{k=1}^\infty \left(\frac{\text{cap}_p(E \cap B_k, 2B_k)}{\text{cap}_p(B_k, 2B_k)} \right)^{1/(p-1)} < \infty.$$

Proof. Since $B(x, 2^{-k-1}) \subset B_k \subset B(x, 2^{-k})$, the claim is an easy consequence of Lemma 4.6 in Björn–Björn–Latvala [6] and Lemma 3.3 in [10] (or Proposition 6.16 in [4]). \square

Proof of Theorem 4.5. Since X is separable there is a countable dense subset $Z \subset X$. Let $\{B_k\}_{k=1}^\infty$ be an enumeration of all open balls with rational radii $< \frac{1}{4} \text{diam } X$ and centres in Z . We define a monotone set function Φ by setting

$$\Phi(E) = \sum_{k=1}^\infty 2^{-k} \frac{\text{cap}_p(E \cap B_k, 2B_k)}{\text{cap}_p(B_k, 2B_k)}$$

for any $E \subset X$. Note that $\Phi(E) \leq 1$, and $C_p(E) = 0$ if and only if $\Phi(E) = 0$.

Let $U := \bigcup_{V \in \mathcal{V}} V$ and define

$$\delta := \inf \left\{ \Phi \left(U \setminus \bigcup_{W \in \mathcal{W}} W \right) : \mathcal{W} \subset \mathcal{V} \text{ is countable} \right\} \leq 1.$$

Then for each $j = 1, 2, \dots$ we may choose a countable subfamily \mathcal{V}'_j of \mathcal{V} such that

$$\Phi \left(U \setminus \bigcup_{V' \in \mathcal{V}'_j} V' \right) \leq \delta + \frac{1}{j}.$$

By defining $\mathcal{V}' := \bigcup_{j=1}^\infty \mathcal{V}'_j$ we then have $\delta = \Phi(F)$ for $F = U \setminus \bigcup_{V' \in \mathcal{V}'} V'$.

It is enough to show that $\delta = 0$. Assume on the contrary that $\delta > 0$. We now invoke the fine Kellogg property from Björn–Björn–Latvala [7, Corollary 1.3] which says that

$$C_p(E \setminus b_p E) = 0 \quad \text{for any set } E \subset X,$$

where the *base* $b_p E$ consists of all points in X at which E is not thin. Since $C_p(F) > 0$, this implies that

$$0 < C_p(F) \leq C_p(F \cap b_p F) + C_p(F \setminus b_p F) = C_p(F \cap b_p F).$$

Accordingly, there exists $x \in F \cap b_p F \subset U$. Choose $V_0 \in \mathcal{V}$ such that $x \in V_0$. Then the set $F \setminus V_0$ is thin at x whereas F is not thin at x . Now Lemma 4.6 implies the existence of a ball B_k with centre in Z and rational radius $< \frac{1}{4} \text{diam } X$ such that

$$\text{cap}_p((F \setminus V_0) \cap B_k, 2B_k) < \text{cap}_p(F \cap B_k, 2B_k).$$

Then we have $\Phi(F \setminus V_0) < \Phi(F) = \delta$, which is a contradiction. Hence the claim holds. \square

Remark 4.7. In the proof of the quasi-Lindelöf principle we used the fine Kellogg property, whose proof depends in turn on the Choquet property, the Cartan property and ultimately on the Cheeger differentiable structure, see Björn–Björn–Latvala [7]. It would be nice if one could obtain a more elementary proof of the fine Kellogg property.

In fact, here we only used a seemingly milder consequence of the fine Kellogg property: if $C_p(F) > 0$, then $F \cap b_p F \neq \emptyset$. However this consequence is equivalent to the full fine Kellogg property, which can be seen as follows: Let $F = E \setminus b_p E$ for some set E . As $F \subset E$, we directly have $b_p F \subset b_p E$, and thus $F \cap b_p F \subset (E \setminus b_p E) \cap b_p E = \emptyset$. Hence the assumption implies that $C_p(E \setminus b_p E) = C_p(F) = 0$.

We can now characterize quasiopen sets in several ways. See Kilpeläinen–Malý [18, Theorem 1.5] for the corresponding result in \mathbf{R}^n .

Proposition 4.8. *Let $U \subset X$ be arbitrary. Then the following conditions are equivalent:*

- (i) U is quasiopen;
- (ii) U is a union of a finely open set and a set of capacity zero;
- (iii) $U = \{x : u(x) > 0\}$ for some nonnegative quasicontinuous u on X ;
- (iv) $U = \{x : u(x) > 0\}$ for some nonnegative $u \in N^{1,p}(X)$.

Proof. (i) \Rightarrow (ii) This is Theorem 4.9 (a) in Björn–Björn–Latvala [6].

(ii) \Rightarrow (i) This follows from Theorem 1.4 (a) in Björn–Björn–Latvala [7].

(iv) \Rightarrow (iii) By Theorem 1.1 in Björn–Björn–Shanmugalingam [9], u is quasicontinuous and thus (iii) holds.

(iii) \Rightarrow (i) This is well known and easy to prove.

(ii) \Rightarrow (iv) Assume that $V \subset U$ is finely open and $C_p(U \setminus V) = 0$. By Lemma 4.4, for each $x \in V$ we find $v_x \in N^{1,p}_0(V)$ such that $0 \leq v_x \leq 1$, $v_x(x) = 1$ and $v_x = 0$ outside of V . Since v_x is quasicontinuous, $\{y : v_x(y) > 0\}$ is a quasiopen subset of V . Therefore, by the quasi-Lindelöf principle (Theorem 4.5), the family $\{v_x : x \in V\}$ contains a countable subfamily $\{v_j\}_{j=1}^\infty$ such that $C_p(Z) = 0$ for the set

$$Z := \{x \in V : v_j(x) = 0 \text{ for all } j\}.$$

Choose $a_j > 0$ so that $v := \sum_{j=1}^\infty a_j v_j \in N^{1,p}(X)$ and let

$$u = \begin{cases} 1 & \text{on } Z \cup (U \setminus V), \\ v & \text{elsewhere.} \end{cases}$$

Since $u = v$ q.e., also $u \in N^{1,p}(X)$. Moreover $U = \{x : u(x) > 0\}$. \square

5. Local Newtonian spaces on quasiopen sets

Solutions of differential equations and variational problems are usually considered in local Sobolev spaces. On quasiopen sets there are at least three different natural candidates for a local Newtonian space. The first is $N_{\text{loc}}^{1,p}(U)$ that we introduced already in Section 2. The others are $N_{\text{fine-loc}}^{1,p}(U)$ and $N_{\text{quasi-loc}}^{1,p}(U)$ that we introduce below.

The space $N_{\text{loc}}^{1,p}(U)$ is natural in connection with Newtonian spaces on metric spaces, but for the classical Sobolev spaces it is a less obvious definition to consider. On \mathbf{R}^n it is not even obvious how to define (nonlocal) Sobolev spaces on quasiopen sets. A fruitful definition of $W^{1,p}(U)$ was given by Kilpeläinen–Malý [18] and they also introduced a local space (called $W_{\text{loc}}^{1,p}(U)$ therein) which is similar to our definition of $N_{\text{fine-loc}}^{1,p}(U)$. The same definitions are used in Latvala [21]. Yet another definition of a local space was considered by Malý–Ziemer [22, p. 149] (and called $W_{p\text{-loc}}^{1,p}(U)$), see also Open problem 7.10. This last definition inspired our definition of $N_{\text{quasi-loc}}^{1,p}(U)$ below. To give it we first need to discuss quasicoverings.

Definition 5.1. A family \mathcal{B} of quasiopen sets is a *quasicovering* of a set E if it is countable, $\bigcup_{U \in \mathcal{B}} U \subset E$ and $C_p(E \setminus \bigcup_{U \in \mathcal{B}} U) = 0$. If every $V \in \mathcal{B}$ is a finely open p -strict subset of E and $\overline{V} \subseteq E$, then \mathcal{B} is a *p -strict quasicovering* of E .

This definition of a quasicovering is slightly different from the one in Kilpeläinen–Malý [18], in that we require the covering to be countable and that it consists of subsets of E . This will be convenient for us, as we have no use for other quasicoverings in this paper. Moreover with our definition we get another characterization of quasiopen sets in Lemma 5.3 below.

Proposition 5.2. *Let U be quasiopen. Then there exists a p -strict quasicovering \mathcal{B} of U consisting of finely open sets. Moreover, the Newtonian functions associated with the p -strict subsets in \mathcal{B} can be chosen compactly supported in U .*

Proof. By Proposition 4.8 we can write $U = V \cup E$, where V is finely open and $C_p(E) = 0$. For every $x \in V$, Lemma 4.4 provides us with a finely open set $V_x \ni x$ such that $V_x \subseteq V$ and the corresponding function $v_x \in N_0^{1,p}(V)$ has compact support in V . The collection $\mathcal{B}' = \{V_x\}_{x \in V}$ covers V and by the quasi-Lindelöf principle (Theorem 4.5), and the fact that $C_p(E) = 0$, there exists a countable subcollection $\mathcal{B} \subset \mathcal{B}'$ such that $C_p(U \setminus \bigcup_{V_x \in \mathcal{B}} V_x) = 0$. \square

Lemma 5.3. *The set E has a quasicovering if and only if it is quasiopen.*

Proof. If E is quasiopen, then obviously $\{E\}$ is a quasicovering of E .

Conversely, if E has a quasicovering $\{U_j\}_{j=1}^\infty$, then $U = \bigcup_{j=1}^\infty U_j$ is quasiopen, by the countable subadditivity of C_p . As $C_p(E \setminus U) = 0$ and C_p is an outer capacity, by Corollary 1.3 in Björn–Björn–Shanmugalingam [9] (or Theorem 5.31 in [4]), also E is quasiopen. \square

We now turn to the different local Newtonian and Dirichlet spaces. Recall that $N_{\text{loc}}^{1,p}(A)$ and $D_{\text{loc}}^p(A)$ were already defined in Section 2.

Definition 5.4. Let A be measurable. We say that

- (i) $u \in N_{\text{fine-loc}}^{1,p}(A)$ if $u \in N^{1,p}(V)$ for all finely open p -strict subsets $V \subseteq A$;
- (ii) $u \in N_{\text{quasi-loc}}^{1,p}(A)$ if there is a quasicovering \mathcal{B} of A such that $u \in N^{1,p}(U)$ for all $U \in \mathcal{B}$.

We similarly define $D_{\text{fine-loc}}^p(A)$, $D_{\text{quasi-loc}}^p(A)$, $L_{\text{fine-loc}}^p(A)$ and $L_{\text{quasi-loc}}^p(A)$.

Note that in view of Lemma 5.3 the definitions of $N_{\text{quasi-loc}}^{1,p}(A)$ and $D_{\text{quasi-loc}}^p(A)$ are useful only when A is quasiopen, otherwise $N_{\text{quasi-loc}}^{1,p}(A) = D_{\text{quasi-loc}}^p(A) = \emptyset$.

Let us take a moment to study these spaces and compare them to $N_{\text{loc}}^{1,p}(A)$ and $D_{\text{loc}}^p(A)$. The following lemma belongs to folklore, but as it is used several times, we state it here and include the short proof. Together with Proposition 4.8 it implies that in the definition of $N_{\text{fine-loc}}^{1,p}(A)$ (or $D_{\text{fine-loc}}^p(A)$) one can equivalently require that $u \in N^{1,p}(U)$ (or $u \in D^p(U)$) for all quasiopen p -strict subsets $U \Subset A$. It can also be seen as a motivation for the definitions of $N_{\text{quasi-loc}}^{1,p}(A)$ and $D_{\text{quasi-loc}}^p(A)$.

Lemma 5.5. *Let $E \subset A$ be measurable sets such that $C_p(A \setminus E) = 0$. If $u : A \rightarrow \overline{\mathbf{R}}$, then $g : A \rightarrow [0, \infty]$ is a p -weak upper gradient of u with respect to A if and only if it is a p -weak upper gradient with respect to E . In particular, $N^{1,p}(A) = N^{1,p}(E)$ with equal norms and equal minimal p -weak upper gradients.*

Similar statements hold for D^p , $N_{\text{quasi-loc}}^{1,p}$ and $D_{\text{quasi-loc}}^p$.

Note that the corresponding statements for $N_{\text{loc}}^{1,p}$, D_{loc}^p , $N_{\text{fine-loc}}^{1,p}$ and $D_{\text{fine-loc}}^p$ are false, as seen by e.g. letting A be the unit ball in \mathbf{R}^n with $n \geq p$ and $E = A \setminus \{0\}$.

Proof. Clearly, $N^{1,p}(A) \subset N^{1,p}(E)$ and every p -weak upper gradient with respect to A is a p -weak upper gradient with respect to E .

Conversely, let g be a p -weak upper gradient of u in E . Proposition 1.48 in [4] shows that $\mu(A \setminus E) = 0$ and that p -almost every curve in A avoids $A \setminus E$. From the latter we immediately see that g is also a p -weak upper gradient in A . As $\mu(A \setminus E) = 0$, the equality of the $N^{1,p}$ -spaces and their norms follows.

Since a quasicovering of E is automatically a quasicovering of A , we immediately have that $N_{\text{quasi-loc}}^{1,p}(E) \subset N_{\text{quasi-loc}}^{1,p}(A)$. Conversely, if $u \in N_{\text{quasi-loc}}^{1,p}(A)$ then there is a quasicovering $\{U_j\}_{j=1}^\infty$ of A such that $u \in N^{1,p}(U_j)$ for each j . As $C_p(U_j \setminus E) = 0$, we see that $U_j \cap E$ is quasiopen, and thus $\{U_j \cap E\}_{j=1}^\infty$ is a quasicovering of E . Since $u \in N^{1,p}(U_j \cap E)$ for all j , we obtain that $u \in N_{\text{quasi-loc}}^{1,p}(E)$.

The proofs for D^p and $D_{\text{quasi-loc}}^p$ are similar. \square

Proposition 5.6. (i) *If A is measurable, then $N_{\text{loc}}^{1,p}(A) \subset N_{\text{fine-loc}}^{1,p}(A)$.*

(ii) *If U is quasiopen, then $N_{\text{loc}}^{1,p}(U) \subset N_{\text{fine-loc}}^{1,p}(U) \subset N_{\text{quasi-loc}}^{1,p}(U)$.*

(iii) *If Ω is open, then $N_{\text{loc}}^{1,p}(\Omega) = N_{\text{fine-loc}}^{1,p}(\Omega) \subset N_{\text{quasi-loc}}^{1,p}(\Omega)$.*

(iv) *If Ω is open and all quasiopen subsets of Ω are open, then*

$$N_{\text{loc}}^{1,p}(\Omega) = N_{\text{fine-loc}}^{1,p}(\Omega) = N_{\text{quasi-loc}}^{1,p}(\Omega). \quad (5.1)$$

Similar statements hold for D^p . Moreover, if Ω is open, then

$$N_{\text{loc}}^{1,p}(\Omega) = N_{\text{fine-loc}}^{1,p}(\Omega) = D_{\text{loc}}^p(\Omega) = D_{\text{fine-loc}}^p(\Omega). \quad (5.2)$$

Note that in general $N_{\text{loc}}^{1,p}(\Omega) \neq N_{\text{quasi-loc}}^{1,p}(\Omega)$ even when Ω is open. Let e.g. Ω be the unit ball B in \mathbf{R}^n with $n \geq p$. Then any function $u \in N_{\text{loc}}^{1,p}(B \setminus \{0\})$ belongs to $N_{\text{quasi-loc}}^{1,p}(B)$, but not all such functions belong to $N_{\text{loc}}^{1,p}(B)$.

Open problem 5.7. Is the first inclusion in (ii) sometimes strict?

Open problem 5.8. Is $N_{\text{loc}}^{1,p}(U) = D_{\text{loc}}^p(U)$ and/or $N_{\text{fine-loc}}^{1,p}(U) = D_{\text{fine-loc}}^p(U)$ if U is quasiopen?

We will later show that $N_{\text{quasi-loc}}^{1,p}(U) = D_{\text{quasi-loc}}^p(U)$ if U is quasiopen, see Corollary 7.6. Our proof of this equality is quite involved, and it would be interesting to know if it can be deduced more easily. (If U is not quasiopen, then $N_{\text{quasi-loc}}^{1,p}(U) = D_{\text{quasi-loc}}^p(U) = \emptyset$.)

Open problem 5.9. Another open problem, related to Proposition 5.6 (iv), is whether all quasiopen sets are open if and only if the capacity of every point is positive. One implication is clear: if $C_p(\{x\}) = 0$, then $\{x\}$ is a quasiopen nonopen set, see the proof below.

A positive answer to this last question would lead to a positive answer to Open problem 5.38 in [4].

Proof of Proposition 5.6. (i) Let $f \in N_{\text{loc}}^{1,p}(A)$ and let $V \Subset A$ be a finely open p -strict subset. For every $x \in \overline{V}$ there is $r_x > 0$ such that $f \in N^{1,p}(B(x, r_x) \cap A)$. Since \overline{V} is compact there is a finite subcover $\{B(x_j, r_{x_j}) \cap A\}_{j=1}^m$ such that $\overline{V} \subset \bigcup_{j=1}^m B(x_j, r_{x_j}) \cap A$. It follows from Lemma 3.3 that $\|f\|_{N^{1,p}(\overline{V})}^p \leq \sum_{j=1}^m \|f\|_{N^{1,p}(B(x_j, r_{x_j}) \cap A)}^p < \infty$, and thus $f \in N^{1,p}(V)$.

(ii) The first inclusion was established in (i), so assume that $f \in N_{\text{fine-loc}}^{1,p}(U)$. By Proposition 5.2, U has a p -strict quasicovering $\{V_j\}_{j=1}^\infty$ consisting of finely open p -strict subsets V_j such that $\overline{V_j} \Subset U$. By assumption, $f \in N^{1,p}(V_j)$ for each j , and thus $f \in N_{\text{quasi-loc}}^{1,p}(U)$.

(iii) Let $f \in N_{\text{fine-loc}}^{1,p}(\Omega)$ and $x \in \Omega$. Then there is r_x such that $B(x, r_x) \Subset \Omega$. It is straightforward to see that $B(x, r_x)$ is a p -strict subset of Ω , and thus $f \in N^{1,p}(B(x, r_x))$. Hence $f \in N_{\text{loc}}^{1,p}(\Omega)$. The remaining inclusions follow from (ii).

(iv) Let $f \in N_{\text{quasi-loc}}^{1,p}(\Omega)$. Then there is a quasicovering $\{U_j\}_{j=1}^\infty$ of Ω so that $f \in N^{1,p}(U_j)$. We shall show that $\Omega = G := \bigcup_{j=1}^\infty U_j$. Assume on the contrary that there is a point $x \in \Omega \setminus G$. As $C_p(\{x\}) \leq C_p(\Omega \setminus G) = 0$ and C_p is an outer capacity (by Björn–Björn–Shanmugalingam [9, Corollary 1.3] or [4, Theorem 5.31]), $\{x\}$ is quasiopen. But since X is connected (which follows from the Poincaré inequality, see Proposition 4.2 in [4]), $\{x\}$ is not open, a contradiction. Hence $\Omega = G$, and thus for each $x \in \Omega$ there is j such that $x \in U_j$ and $f \in N^{1,p}(U_j)$. By assumption U_j is open, and hence $f \in N_{\text{loc}}^{1,p}(\Omega)$. The other inclusions in (5.1) follow from (iii).

The corresponding results for D^p are proved similarly. Finally, if Ω is open, then $N_{\text{loc}}^{1,p}(\Omega) = D_{\text{loc}}^p(\Omega)$, by Proposition 4.14 and Corollary 4.24 in [4], and hence (5.2) follows from (iii) and the corresponding result for D^p . \square

Example 5.10. One may ask if the requirement $V \Subset A$ in the definition of $N_{\text{fine-loc}}^{1,p}(A)$ is essential, or stated in another way: If $u \in N_{\text{fine-loc}}^{1,p}(A)$ and V is a finely open p -strict subset of A , does it then follow that $u \in N^{1,p}(V)$? This may fail even for open $A = \Omega$ and open p -strict subsets V of Ω . Indeed, let $1 < p < 2$,

$$\Omega = (-1, 1) \times (0, 1) \subset \mathbf{R}^2 \quad \text{and} \quad V = \{(x_1, x_2) \in \Omega : |x_1| < x_2 < \tfrac{1}{6}\}.$$

Then V is an open p -strict subset of Ω , by Example 5.7 in Björn–Björn [3] (or Example 11.10 in [4]) as f therein belongs to $N_0^{1,p}(\Omega)$ and $f = 1$ on V . Let $u(x) = |x|^\beta$ with $\beta \leq -2/p$. Then, $u \in N_{\text{loc}}^{1,p}(\Omega) = N_{\text{fine-loc}}^{1,p}(\Omega)$ (cf. Proposition 5.6 (iii)), but $u \notin L^p(V)$ and thus $u \notin N^{1,p}(V)$.

On a finely open set V yet another local Newtonian space may be considered: $u \in N_{\text{f-loc}}^{1,p}(V)$ if for every $x \in V$ there is a finely open set $V_x \subset V$ such that $x \in V_x$ and $u \in N^{1,p}(V_x)$. By Lemma 4.4, one can equivalently require that $V_x \Subset V$. Lemma 4.4 and the quasi-Lindelöf principle (Theorem 4.5) immediately imply that

$$N_{\text{fine-loc}}^{1,p}(V) \subset N_{\text{f-loc}}^{1,p}(V) \subset N_{\text{quasi-loc}}^{1,p}(V). \quad (5.3)$$

It is natural to ask if these inclusions can be strict. Indeed this is so even if V is open, as we show in the following examples.

Example 5.11. To see that the first inclusion in (5.3) can be strict, let $V = B$ be the unit ball in \mathbf{R}^3 , $p = 2$ and let

$$E = \{(x, t) \in \mathbf{R}^2 \times \mathbf{R} : t > 0 \text{ and } |x| < e^{-1/t}\}$$

be the Lebesgue spine, which is thin at the origin (see e.g. Example 13.4 in [4]). Let

$$u(x, t) = \left(\frac{e^{2/t}}{t}\right)^{1/p} \varphi(x, t),$$

where $\varphi \in C^\infty(\mathbf{R}^3 \setminus \{0\})$ is such that

$$\varphi(x, t) = \begin{cases} 1, & \text{if } |x| < \frac{1}{3}e^{-1/t}, \\ 0, & \text{if } |x| > \frac{2}{3}e^{-1/t}, \end{cases}$$

and we define u arbitrarily at the origin.

Then $u \in N_{\text{loc}}^{1,p}(\mathbf{R}^3 \setminus \{0\})$ and it is easily verified that $u \notin L^p(B)$. As $C_p(\{0\}) = 0$, $u = 0$ in $B \setminus \overline{E}$ and $(B \setminus \overline{E}) \cup \{0\}$ is a finely open set containing 0, it is still true that $u \in N_{\text{f-loc}}^{1,p}(B)$, but $u \notin N_{\text{loc}}^{1,p}(B) = N_{\text{fine-loc}}^{1,p}(B)$ (by Proposition 5.6 (iii)).

Example 5.12. To see that the last inclusion in (5.3) can be strict, we let $V = B$ be the unit ball in \mathbf{R}^n with $1 < p \leq n$, and let $u(x) = |x|^\beta$ with $\beta \leq -n/p$. Let V_0 be any finely open set containing 0. Then 0 is a density point of V_0 , by Corollary 2.51 in Malý–Ziemer [22]. Thus there are $r_0, \theta > 0$ such that for all $0 < r < r_0$,

$$\Lambda_1(\{\rho : 0 < \rho < r \text{ and } \Lambda_{n-1}(\{x \in V_0 : |x| = \rho\}) > \theta \rho^{n-1}\}) > \frac{1}{2}r, \quad (5.4)$$

where Λ_d denotes d -dimensional Lebesgue measure. Using polar coordinates it is easy to see that $u \notin L^p(B)$. It then follows from (5.4) that $u \notin L^p(V_0)$, and hence $u \notin N_{\text{f-loc}}^{1,p}(B)$. On the other hand, as $C_p(\{0\}) = 0$, we see that $u \in N_{\text{loc}}^{1,p}(B \setminus \{0\}) \subset N_{\text{quasi-loc}}^{1,p}(B \setminus \{0\}) = N_{\text{quasi-loc}}^{1,p}(B)$, by Lemma 5.5.

6. Quasicontinuity

Our aim in this section is to prove Theorem 1.2. In fact, we prove the following generalization for local Dirichlet spaces.

Theorem 6.1. *Let U be quasiopen and assume that $u \in D_{\text{quasi-loc}}^p(U)$. Then u is finite q.e. and finely continuous q.e. in U . In particular, u is quasicontinuous in U .*

For open U , quasicontinuity was deduced in Björn–Björn–Shanmugalingam [9, Theorem 1.1] for $u \in N_{\text{loc}}^{1,p}(\Omega)$. To prove Theorem 6.1 we will use the following two lemmas. See Björn–Björn–Latvala [7] for a proof of the first one.

Lemma 6.2. *Let Y be a metric space equipped with a measure which is bounded on balls. If $u, v \in N^{1,p}(Y)$ are bounded, then $uv \in N^{1,p}(Y)$.*

Lemma 6.3. *Let U be quasiopen and $u \in D_{\text{quasi-loc}}^p(U)$. Then u is finite q.e. in U .*

Proof. By assumption there is a quasicovering $\mathcal{B} = \{U_j\}_{j=1}^\infty$ of U such that $u \in N^{1,p}(U_j)$ for each j . By Lemma 3.2 and Proposition 3.5, u is finite q.e. in U_j . Since \mathcal{B} is a quasicovering, it follows that u is finite q.e. in U . \square

Proof of Theorem 6.1. By assumption there is a quasicovering \mathcal{B} of U such that $u \in D^p(\tilde{U})$ for each $\tilde{U} \in \mathcal{B}$. In addition we can assume that all $\tilde{U} \in \mathcal{B}$ are bounded. By Proposition 5.2, for each $\tilde{U} \in \mathcal{B}$ there exists a p -strict quasicovering $\mathcal{B}_{\tilde{U}}$ of \tilde{U} consisting of finely open p -strict subsets $V_{j,\tilde{U}}$ such that $\overline{V_{j,\tilde{U}}} \Subset \tilde{U}$.

First, assume that u is bounded and let $V = V_{j,\tilde{U}}$ be arbitrary. Then there is $v \in N_0^{1,p}(\tilde{U})$ with $v = 1$ on V and $0 \leq v \leq 1$ everywhere. Since u and \tilde{U} are bounded and $u \in D^p(\tilde{U})$, we get that $u \in N^{1,p}(\tilde{U})$. Let $w = vu$, extended by 0 outside of \tilde{U} . Then $w \in N^{1,p}(\tilde{U})$ by Lemma 6.2. As $|w| \leq Cv \in N_0^{1,p}(\tilde{U})$ (with $C = \sup_U |u| < \infty$), it follows from Lemma 2.37 in [4] that $w \in N_0^{1,p}(\tilde{U}) \subset N^{1,p}(X)$, and in particular w is quasicontinuous in X . By Theorem 4.9 (b) in Björn–Björn–Latvala [6], w is finely continuous q.e. in X . Since $u = w$ in the finely open set V , it follows that u is finely continuous q.e. in V , and thus q.e. in U , as V was arbitrary.

If u is arbitrary, let $u_k = \max\{\min\{u, k\}, -k\}$, $k = 1, 2, \dots$, be the truncations of u at levels $\pm k$. By the first part of the proof there is a set E_k such that $C_p(E_k) = 0$ and u_k is finely continuous at all $x \in U \setminus E_k$.

By Lemma 6.3, there is a set E_0 with $C_p(E_0) = 0$ such that u is finite in $U \setminus E_0$. Let $E = \bigcup_{j=0}^{\infty} E_j$, which is a set with zero capacity. If $x \in U \setminus E$, then $u(x)$ is finite and hence there is k such that $|u(x)| < k$. Since u_k is finely continuous at x and $|u(x)| < k$, we conclude that u is also finely continuous at x . Hence u is finely continuous q.e. in U .

The quasicontinuity of u now follows from Theorem 1.4 (b) in Björn–Björn–Latvala [7]. \square

Proof of Theorem 1.2. This follows directly from Proposition 5.6 (ii) together with Theorem 6.1. \square

7. Sobolev spaces based on fine upper gradients

In this section we assume that U is quasiopen.

The main aim of this section is to prove Theorem 1.1. To do so it will be convenient to make the following definition, for reasons that will become clear towards the end of the section. It has been inspired by the fine gradients in \mathbf{R}^n from Kilpeläinen–Malý [18], see Definition 7.7 below and the comments after it. Recall that $g_{u,V}$ is the minimal p -weak upper gradient of $u : V \rightarrow \overline{\mathbf{R}}$ taken with respect to V as the ambient space.

Definition 7.1. A function $\tilde{g}_u : U \rightarrow [0, \infty]$ is a p -fine upper gradient of $u \in D_{\text{quasi-loc}}^p(U)$ if there is a quasicovering \mathcal{B} of U such that $u \in D^p(V)$ for every $V \in \mathcal{B}$ and $\tilde{g}_u = g_{u,V}$ a.e. in V .

The following result shows that p -fine upper gradients always exist.

Lemma 7.2. *If $u \in D_{\text{quasi-loc}}^p(U)$, then it has a unique p -fine upper gradient \tilde{g}_u .*

The uniqueness is up to a.e. Moreover, by Definition 7.1, if $g : U \rightarrow [0, \infty]$ satisfies $g = \tilde{g}_u$ a.e., then g is also a p -fine upper gradient of u .

Proof. Let \mathcal{B} be a quasicovering of U such that $u \in D^p(V)$ for every $V \in \mathcal{B}$. If $V, W \in \mathcal{B}$, then $V \cap W$ is quasiopen and Lemma 3.3 shows that

$$g_{u,V} = g_{u,V \cap W} = g_{u,W} \quad \text{a.e. in } V \cap W.$$

We can therefore define $\tilde{g}_u : U \rightarrow [0, \infty]$ so that $\tilde{g}_u = g_{u,V}$ a.e. in V for every $V \in \mathcal{B}$. By definition, \tilde{g}_u is a p -fine upper gradient of u .

To prove the uniqueness, assume that g is any p -fine upper gradient of u , and let \mathcal{B} and \mathcal{B}' be the quasicoverings given in Definition 7.1 for \tilde{g}_u and g , respectively. Let $V \in \mathcal{B}$ and $W \in \mathcal{B}'$. Since $V \cap W$ is quasiopen, Lemma 3.3 then yields that

$$\tilde{g}_u = g_{u,V} = g_{u,V \cap W} = g_{u,W} = g \quad \text{a.e. in } V \cap W.$$

As \mathcal{B}' and \mathcal{B} are quasicoverings it follows that $\tilde{g}_u = g$ a.e. in V , and thus in U . This proves the a.e. uniqueness of \tilde{g}_u . \square

Next, we show that p -fine upper gradients are the same as minimal p -weak upper gradients, with minimality in the appropriate sense. Note that minimality has been built into the definition of p -fine upper gradients.

Theorem 7.3. *Let $u \in D_{\text{quasi-loc}}^p(U)$ and let \tilde{g}_u be a p -fine upper gradient of u . Then $\tilde{g}_u \in L_{\text{quasi-loc}}^p(U)$ and it is a p -weak upper gradient of u in U which is minimal in the following two senses:*

- (a) *If $W \subset U$ is quasiopen and $u \in D_{\text{loc}}^p(W)$, then $\tilde{g}_u = g_{u,W}$ a.e. in W .*
- (b) *If $g \in L_{\text{quasi-loc}}^p(U)$ is a p -weak upper gradient of u , then $\tilde{g}_u \leq g$ a.e.*

Proof. Let \mathcal{B} be the quasicovering of U associated with \tilde{g}_u in Definition 7.1. It is immediate that $\tilde{g}_u \in L_{\text{quasi-loc}}^p(U)$.

Next we show that \tilde{g}_u is a p -weak upper gradient of u in U . Since $C_p(U \setminus \bigcup_{V \in \mathcal{B}} V) = 0$ and each V is p -path open (by Lemma 3.2), we conclude from Proposition 1.48 in [4] that p -almost every curve γ in U avoids $U \setminus \bigcup_{V \in \mathcal{B}} V$ and is such that $\gamma^{-1}(V)$ is open for each $V \in \mathcal{B}$. Moreover, by [4, Lemma 1.40], we can assume that \tilde{g}_u is an upper gradient of u (i.e. satisfies (2.1)) on any subcurve $\tilde{\gamma} \subset \gamma$, whose image is contained in V . Let $\gamma : [0, l_\gamma] \rightarrow \bigcup_{V \in \mathcal{B}} V$ be such a curve.

Since each $\gamma^{-1}(V)$ is open, it is a countable union of (relatively) open subintervals of $[0, l_\gamma]$, whose collection for all $V \in \mathcal{B}$ covers $[0, l_\gamma]$. By compactness, $[0, l_\gamma]$ can be covered by finitely many such intervals. We can then find slightly smaller closed intervals so that $[0, l_\gamma] = \bigcup_{j=1}^N [a_j, b_j]$, where $[a_j, b_j] \subset \gamma^{-1}(V_j)$ for some $V_j \in \mathcal{B}$. Since this is a finite union of intervals we can, by decreasing their lengths, assume that $0 = a_1 < b_1 = a_2 < \dots < b_N = l_\gamma$. We then have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \sum_{j=1}^N |u(\gamma(a_j)) - u(\gamma(b_j))| \leq \sum_{j=1}^N \int_{\gamma|_{[a_j, b_j]}} \tilde{g}_u \, ds = \int_\gamma \tilde{g}_u \, ds.$$

This shows that \tilde{g}_u is a p -weak upper gradient of u in U .

We now turn to the minimality.

- (a) If $V \in \mathcal{B}$, then $V \cap W$ is quasiopen. Hence by Lemma 3.3,

$$g_{u,W} = g_{u,V \cap W} = g_{u,V} = \tilde{g}_u \quad \text{a.e. in } V \cap W.$$

As \mathcal{B} is a quasicovering, we see that $g_{u,W} = \tilde{g}_u$ a.e. in W .

- (b) Let $\tilde{\mathcal{B}}$ be a quasicovering such that $g \in L^p(\tilde{V})$ for each $\tilde{V} \in \tilde{\mathcal{B}}$. Then $u \in D^p(\tilde{V})$ and by (a),

$$\tilde{g}_u = g_{u,\tilde{V}} \leq g \quad \text{a.e. in } \tilde{V}.$$

As $\tilde{\mathcal{B}}$ is a quasicovering, it follows that $\tilde{g}_u \leq g$ a.e. in U . \square

Theorem 7.4. *A function u belongs to $N^{1,p}(U)$ if and only if $u \in L^p(U)$ and there exists a p -fine upper gradient $\tilde{g}_u \in L^p(U)$. Moreover, in this case, $\tilde{g}_u = g_{u,U}$ a.e. in U .*

Similar statements for $N_{\text{loc}}^{1,p}$, $N_{\text{fine-loc}}^{1,p}$, $N_{\text{quasi-loc}}^{1,p}$, D^p , D_{loc}^p , $D_{\text{fine-loc}}^p$ and $D_{\text{quasi-loc}}^p$ are also valid.

Proof. If $u \in N^{1,p}(U)$, then $u \in L^p(U)$ and the minimal p -weak upper gradient $g_{u,U} \in L^p(U)$. As $u \in N^{1,p}(U) \subset D_{\text{quasi-loc}}^p(U)$ it has a p -fine upper gradient \tilde{g}_u , by Lemma 7.2. By Theorem 7.3, $\tilde{g}_u = g_{u,U}$ a.e., and thus $\tilde{g}_u \in L^p(U)$.

Conversely, if $u \in L^p(U)$ and $\tilde{g}_u \in L^p(U)$ is a p -fine upper gradient of u , then \tilde{g}_u is a p -weak upper gradient of u in U , by Theorem 7.3, and thus $u \in N^{1,p}(U)$. \square

The following result shows that in the definition of p -fine upper gradients, u can be extended from V to a global Newtonian function.

Proposition 7.5. *If $u \in D_{\text{quasi-loc}}^p(U)$, then there is a p -strict quasicovering \mathcal{B} of U such that for every $V \in \mathcal{B}$ there exists $u_V \in N^{1,p}(X)$ with $u = u_V$ in V .*

The following result is an immediate corollary.

Corollary 7.6. $N_{\text{quasi-loc}}^{1,p}(U) = D_{\text{quasi-loc}}^p(U)$.

Proof of Proposition 7.5. Let $\tilde{\mathcal{B}}$ be a quasicovering of U such that $u \in D^p(\tilde{V})$ for every $\tilde{V} \in \tilde{\mathcal{B}}$. Theorem 6.1 shows that u is quasicontinuous. Let $U_k = \{x \in U : |u(x)| < k\}$, $k = 1, 2, \dots$. By quasicontinuity, each U_k is quasiopen and $\{U_k\}_{k=1}^\infty$ forms a quasicovering of U . For each k , let $\mathcal{B}_k = \{U_k \cap \tilde{V} : \tilde{V} \in \tilde{\mathcal{B}}\}$ and for each $W \in \mathcal{B}_k$ use Proposition 5.2 to obtain a p -strict quasicovering \mathcal{B}_W of W , together with the associated compactly supported functions in W . Then $\mathcal{B} = \bigcup_{k=1}^\infty \bigcup_{W \in \mathcal{B}_k} \mathcal{B}_W$ is a p -strict quasicovering of U , and u is bounded on each $W \in \mathcal{B}$.

Now, let k , $W = U_k \cap \tilde{V} \in \mathcal{B}_k$ and $V \in \mathcal{B}_W$ be arbitrary, and let $v \in N_0^{1,p}(W)$ be the associated function such that $v = 1$ in V . Since $u \in D^p(W)$ is bounded in W and v has compact support in W , we see that $u \in N^{1,p}(\text{supp } v)$. Lemma 6.2 implies that $uv \in N^{1,p}(\text{supp } v)$, and as $|uv| \leq kv \in N_0^{1,p}(\text{supp } v)$, Lemma 2.37 in [4] then shows that $uv \in N_0^{1,p}(\text{supp } v) \subset N^{1,p}(X)$. Let $u_V = uv \in N^{1,p}(X)$ for $V \in \mathcal{B}$. Then $u = u_V$ in V . \square

The following definition is from Kilpeläinen–Malý [18], and has been our inspiration for Definition 7.1. Note however, that the metric space theory allows us to consider $u \in D^p(V)$ in Definition 7.1, which makes the situation simpler. In particular, we do not need to go outside of U to define p -fine upper gradients. On the other hand, Proposition 7.5 shows that one can equivalently use functions $u_V \in N^{1,p}(X)$ such that $u = u_V$ in V . (In [18] a quasicovering may be uncountable, but it is required to contain a countable quasicovering in our sense. For the purpose of the definition below the existence of a quasicovering in either sense is obviously equivalent.)

Definition 7.7. Let $U \subset \mathbf{R}^n$ be quasiopen. A function $u : U \rightarrow \mathbf{R}$ belongs to $W^{1,p}(U)$ if

- (i) $u \in L^p(U)$;
- (ii) there is a quasicovering \mathcal{B} of U such that for every $V \in \mathcal{B}$ there is an open set $G_V \supset V$ and $u_V \in W^{1,p}(G_V)$ so that $u = u_V$ in V ;
- (iii) the *fine gradient* ∇u , defined by $\nabla u = \nabla u_V$ a.e. on each $V \in \mathcal{B}$, also belongs to $L^p(U)$.

Moreover, we let

$$\|u\|_{W^{1,p}(U)} = \left(\int_U (|u|^p + |\nabla u|^p) dx \right)^{1/p}.$$

As \mathcal{B} is a quasicovering, the gradient ∇u is well defined a.e., and we may pick any representative. By Proposition 7.5 and Theorem 7.8 below, one can equivalently require that $G_V = \mathbf{R}^n$ in Definition 7.7.

Kilpeläinen–Malý [18] gave this definition for unweighted \mathbf{R}^n , but it makes sense also in weighted \mathbf{R}^n (with a p -admissible weight), provided that we by ∇u_V mean the corresponding weighted Sobolev gradient, cf. the discussion on p. 13 in Heinonen–Kilpeläinen–Martio [14]. Theorem 7.8 below also holds, with the proof below, for weighted \mathbf{R}^n .

If we change a function in $W^{1,p}(U)$ on a set of measure zero it remains in $W^{1,p}(U)$, in contrast to the Newtonian case. To characterize $W^{1,p}(U)$ using Newtonian spaces we need to consider the space

$$\widehat{N}^{1,p}(U) = \{u : u = v \text{ a.e. for some } v \in N^{1,p}(U)\}. \quad (7.1)$$

We also let $\|u\|_{\widehat{N}^{1,p}(U)} = \|v\|_{N^{1,p}(U)}$, which is well defined by Lemma 1.62 in [4].

Theorem 7.8. *Let $U \subset \mathbf{R}^n$ be a quasiopen set. Then $\widehat{N}^{1,p}(U) = W^{1,p}(U)$ and $\|u\|_{\widehat{N}^{1,p}(U)} = \|u\|_{W^{1,p}(U)}$ for all $u \in \widehat{N}^{1,p}(U)$.*

Moreover, $g_{v,U} = |\nabla v|$ a.e. in U for all $u \in \widehat{N}^{1,p}(U)$ and v as in (7.1).

By Remark 6.12 in [4] the Sobolev capacity considered here is the same as in [14] and [18].

Proof. Let $u \in \widehat{N}^{1,p}(U)$. Then there is $v \in N^{1,p}(U)$ such that $v = u$ a.e. in U . By Proposition 7.5, there is a quasicovering \mathcal{B} of U such that for every $V \in \mathcal{B}$ there exists $v_V \in N^{1,p}(\mathbf{R}^n)$ with $v = v_V$ in V . By Proposition A.12 in [4], $v_V \in W^{1,p}(\mathbf{R}^n)$ and $g_{v_V} = |\nabla v_V|$ a.e. in \mathbf{R}^n . Hence $v \in W^{1,p}(U)$ and $\tilde{g}_v = |\nabla v|$ a.e. in U , where \tilde{g}_v and ∇v are the p -fine and fine gradients of v , respectively. Theorem 7.4 implies that $g_{v,U} = \tilde{g}_v = |\nabla v|$ a.e. in U . As $u = v$ a.e., it follows directly that $u \in W^{1,p}(U)$ and $\nabla u = \nabla v$ a.e. Thus,

$$\|u\|_{\widehat{N}^{1,p}(U)} = \|v\|_{N^{1,p}(U)} = \|v\|_{W^{1,p}(U)} = \|u\|_{W^{1,p}(U)}.$$

Conversely, let $u \in W^{1,p}(U)$. Then there is a quasicovering \mathcal{B} with u_V and G_V for $V \in \mathcal{B}$, as in Definition 7.7. By Theorem 4.4 in Heinonen–Kilpeläinen–Martio [14], for each $V \in \mathcal{B}$ there is a quasicontinuous function \tilde{u}_V on the open set G_V such that $\tilde{u}_V = u_V$ a.e. in G_V . By Proposition A.13 in [4], $\tilde{u}_V \in N^{1,p}(G_V)$.

If $V, W \in \mathcal{B}$, then $\tilde{u}_V = \tilde{u}_W$ q.e. in $G_V \cap G_W$, by Kilpeläinen [17] (or [4, Proposition 5.23]). Hence we can find functions $\tilde{u} : U \rightarrow \overline{\mathbf{R}}$ and $g : U \rightarrow [0, \infty]$ such that

$$\tilde{u} = \tilde{u}_V \text{ q.e. in } V \quad \text{and} \quad g = g_{\tilde{u}_V, V} \text{ a.e. in } V$$

for every $V \in \mathcal{B}$. By definition, g is a p -fine upper gradient of \tilde{u} . Lemma 3.3 above and Proposition A.12 in [4] yield that

$$g = g_{\tilde{u}_V, V} = g_{\tilde{u}_V, G_V} = |\nabla \tilde{u}_V| = |\nabla u_V| = |\nabla u| \quad \text{a.e. in } V$$

for every $V \in \mathcal{B}$, and hence $g = |\nabla u|$ a.e. in U . Since $|\nabla u| \in L^p(U)$, Theorem 7.4 implies that $\tilde{u} \in N^{1,p}(U)$. As $u = \tilde{u}$ a.e. in U , we get that $u \in \widehat{N}^{1,p}(U)$. \square

The following gives a more precise description of which representatives of $W^{1,p}(U)$ belong to $N^{1,p}(U)$ (and not just to $\widehat{N}^{1,p}(U)$). As it is valid in metric spaces we state it using $\widehat{N}^{1,p}(U)$, but in view of Theorem 7.8 one can of course replace $\widehat{N}^{1,p}(U)$ by $W^{1,p}(U)$ if $U \subset \mathbf{R}^n$.

Theorem 7.9. *Let $u \in \widehat{N}^{1,p}(U)$. Then the following are equivalent:*

- (a) $u \in N^{1,p}(U)$;
- (b) u is quasicontinuous;
- (c) u is absolutely continuous on p -almost every curve in U .

Proof. (a) \Rightarrow (b) This follows from Theorem 6.1.

(b) \Rightarrow (a) As $u \in \widehat{N}^{1,p}(U)$ there is $v \in N^{1,p}(U)$ such that $v = u$ a.e. in U . By Theorem 6.1, v is quasicontinuous. Hence by Kilpeläinen [17] (or [4, Proposition 5.23]) $u = v$ q.e., and thus also $u \in N^{1,p}(U)$. Here we also need to appeal to Proposition 5.22 in [4], which shows that our capacity satisfies Kilpeläinen's axioms.

(a) \Leftrightarrow (c) This follows from Proposition 1.63 in [4]. \square

Proof of Theorem 1.1. This follows directly from Theorems 7.8 and 7.9. \square

Let us end by stating the following question.

Open problem 7.10. Let $V \subset \mathbf{R}^n$ be finely open. Is then $u \in N_{\text{quasi-loc}}^{1,p}(V)$ if and only if u is quasicontinuous and $u \in W_{p\text{-loc}}^{1,p}(V)$, as defined by Malý–Ziemer [22, p. 149]?

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